Dynamical systems, filtrations, and entropy
by
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SUPPLEMENTARY MANUAL FOR RECORDING OF A LECTURE ON

DYNAMICAL SYSTEMS, FILTRATIONS, AND ENTROPY

by

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M is a C^∞ compact manifold without boundary; $\dim M = m$.

$\mathrm{Diff}^r(M)$ is the set of C^r diffeomorphisms of M.

Problem A: *Genericity*. Find a reasonable generic set of diffeomorphisms.

Problem B: *Model making*. Produce the simplest diffeomorphism in an isotopy class. For "simplest" we require:

a) structural stability,

b) minimal entropy.

Definition 1. A diffeomorphism $f \in \mathrm{Diff}^r(M)$ is *structurally stable* if there exists a set $U_f \subset \mathrm{Diff}^r(M)$ such that for each $g \in U_f$ there is a homeomorphism $h \colon M \to M$ for which $hf = gh$.

Definition 2. Let T be a continuous map of a compact metric space $(X, d) = X$. A subset $E \subset X$ is *(n, ϵ)-separated* if for any $x, y \in E, x \neq y$, there is a j, $0 \leq j \leq n$, such that $d(T^j x, T^j y) > \epsilon$. Let $r_n(\epsilon)$ denote the largest cardinality of any (n, ϵ)-separated set, and define

$$r_\epsilon(T) = \limsup \frac{1}{n} \ln r_n(\epsilon).$$

Then *the topological entropy of* T is defined as

$$h(T) = \lim_{\epsilon \to 0} r_\epsilon(T).$$

$$(1) \qquad \Omega(f) = \{x \in M \mid \text{for any neighborhood } U \text{ of } x \ \exists n > 0 \text{ such that } f^n(U) \cap U \neq \emptyset\}.$$

Definition 3. An *ω-limit point* of x is any point y such that there exist $n_i \to \infty$ with $f^{n_i} x = y$.

Definition 4. A *filtration* $\mathfrak{M}$ for f is a sequence $\phi \subset M_0 \subset \cdots \subset M_k = M$ of compact submanifolds with boundary, $\dim M_i = m$ for all i, such that $f(M_i) \subset \mathrm{Int}\, M_i$.

$$(2) \qquad K_\alpha(\mathfrak{M}) = \bigcap_{n \in \mathbb{Z}} f^n(\overline{M_\alpha - M_{\alpha-1}}),\ K(\mathfrak{M}) = \bigcup_{\alpha=0}^{k} K_\alpha(\mathfrak{M}).$$

Problem 1. Given a finite amount of data about M_α, $M_{\alpha-1}$ and f, compute $\check{H}^*(K_\alpha(\mathfrak{M}))$.

Definition 5. A filtration is called *fine* if $K(\mathfrak{M}) = \Omega$. A filtration $\mathfrak{N}$ *refines* $\mathfrak{M}$ if for each β there exists an α such that $\overline{N_\beta - N_{\beta-1}} \subset \overline{M_\alpha - M_{\alpha-1}}$. A sequence of filtrations $\mathfrak{M}^i$ is called *fine* if each $\mathfrak{M}^{i+1}$ refines $\mathfrak{M}^i$ and $\bigcap_i K(M^i) = \Omega$.

Definition 6. A diffeomorphism $f \in \mathrm{Diff}^r(M)$ *admits no* C^0–Ω *explosions* if for any fixed neighborhood $U_{\Omega(f)}$ there exists a neighborhood $U_f \subset \mathrm{Diff}^r(M)$ (with the C^0 topology) such that $\Omega(g) \subset U_{\Omega(f)}$ for all $g \in U_f$.

Theorem 1 (Shub and Smale). *Let* $m \geq 3$. *Then a diffeomorphism* $f \in \mathrm{Diff}^r(M)$ *has a fine sequence of filtrations if and only if* f *admits no* C^0–Ω *explosions.*

Conjecture 1. The diffeomorphisms which have fine sequences of filtrations are generic.

Definition 7. Let X be a vector field on M. A function $L\colon M \to \mathbf{R}$ is a *Ljapunov function* for X if $X(L) < 0$ on $M - \Omega$ and $DL = 0$ on Ω.

Conjecture 2. A generic set of vector fields has Ljapunov functions of class C^m.

Theorem 2 (Smale). *Every diffeomorphism is isotopic to a structurally stable diffeomorphism.*

$$\phi \subset H_0 \subset \cdots \subset H_m = M, \text{ where } \overline{H_j - H_{j-1}} = \bigcup_{i=1}^{n_j} D_i^j \times D_i^{m-j}. \tag{3}$$

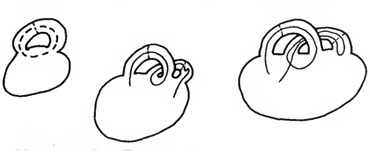

Figure 1

Dashed lines are transverse discs

$$(a_{ij})_{n_j \times n_j} = A_j, \qquad (i_{lk})_{n_j \times n_j} = I_j. \tag{4}$$

algebraic intersections   geometric intersections

Of course $|a_{ij}| \leq i_{ij}$. Let H denote the subset of $\mathrm{Diff}^r(M)$ constructed in this way.

Theorem 3. *H is dense in* $\mathrm{Diff}^r(M)$ *with the* C^0 *topology.*

Question: What is the entropy of elements of H?

Theorem 4. *If* $f \in H$, *then* $h(f) = \max \ln |\lambda|$, *where* λ *is an eigenvalue of any* I_j.

Theorem 5. *If* $f \in H$, *then*

$$(*) \qquad h(f) \geq \max \ln |\lambda|,$$

where λ *is now an eigenvalue of* $f_*\colon H_*(M, \mathbf{R}) \supset$.

Theorem 6. *Let* $f\colon M \to M$ *be continuous and let* $Z^k \subset H^1(M)$ *be invariant for the induced map* f^*. *Suppose there are* k *elements* $e_1, \ldots, e_k \in Z^k$ *such that* $e_1 \cup \cdots \cup e_k \neq 0$. *Finally, suppose that* $f^*\colon Z^k \to Z^k$ *has no eigenvalue of modulus one. Then* $h(f) \geq \Sigma \ln |\lambda|$, *where the sum is taken over all eigenvalues* λ *of* f^* *with* $|\lambda| > 1$.

Problem 2. Is inequality $(*)$ (Theorem 5) true for all structurally stable diffeomorphisms? for all diffeomorphisms?

Problem 3. Can equality be achieved in $(*)$ by an isotopic diffeomorphism which is structurally stable? which belongs to H?

Definition 8. A map $f \in \mathrm{Diff}^r(M)$ is a *Morse-Smale diffeomorphism* if it is structurally stable and Ω is finite.

Proposition 1 (Bowen). $h(f) = h(f \mid \Omega)$; *e.g.* $h(f) = 0$ *for finite* Ω.

Problem 4. If f is a structurally stable diffeomorphism with $h(f) = 0$, does it follow that f is a Morse-Smale diffeomorphism?

Theorem 7. *If* f *is a Morse-Smale diffeomorphism, then the induced map* $f_*\colon H_1(M, \mathbf{R}) \supset$ *is quasi-unipotent.*

Problem 5. Do all Morse-Smale diffeomorphisms belong to H?

Theorem 8 (Shub and Sullivan).(1) *If* $\Pi_1(M) = 0$, $\dim M \geq 6$ *and* $f_*\colon H_*(M, \mathbf{R}) \supset$ *is quasi-unipotent, then* f *is isotopic to a Morse-Smale diffeomorphism in* H.

Thus we have solved Problem B for $\max \ln |\lambda| = 0$. Now consider the case where $\max \ln |\lambda| > 0$.

Theorem 9. *There is an isotopy class* $\mathcal{I}$ *in* $\mathrm{Diff}^r(M)$, *where*

$$M = S^3 \times S^3 \,\#\, S^3 \times S^3 \,\#\, S^3 \times S^3 \,\#\, S^3 \times S^3,$$

such that $h(f) > \max \ln |\lambda|$ *for any* $f \in \mathcal{I} \cap H$.

Problem 6. Given an isotopy class $\mathcal{I}$, does there exist a sequence $\{f_n\}$, $f_i \in \mathcal{I} \cap H$, such that $h(f_n) \to \max \ln |\lambda|$?

Problem 7. Given a structurally stable diffeomorphism $g \in \mathcal{I}$, does there exist an $f \in \mathcal{I} \cap H$ such that $h(f) < h(g)$?

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(1) *Added in proof.* Some problems have developed in writing down the proof of Theorem 8, and it is not now clear whether it is true in the generality in which it is stated.

$$K = \begin{pmatrix} \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} & & & \\ & \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} & & \mathbf{0} \\ \mathbf{0} & & \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} & \\ & & & \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \end{pmatrix}, \qquad K' = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

(5)

$$A' = \begin{pmatrix} B & 0 \\ 0 & (B^{-1})^t \end{pmatrix}, \qquad A'^t K A' = K.$$

Let B be the companion matrix of $x^4 + x + 1$ (not all roots of this polynomial are a real number times a root of unity). Then A represents the class we wanted.

(6)

$$\begin{array}{ccc} C_n & \xrightarrow{A} & C_n \\ \downarrow & & \downarrow \\ H_n & \xrightarrow{f_*} & H_n \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array}$$

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